

On topological obstructions to global stabilization of an inverted pendulum

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Abstract

We consider a classical problem of control of an inverted pendulum by means of a horizontal motion of its pivot point. We suppose that the control law can be non-autonomous and non-periodic w.r.t. the position of the pendulum. It is shown that global stabilization of the vertical upward position of the pendulum cannot be obtained for any Lipschitz control law, provided some natural assumptions. Moreover, we show that there always exists a solution separated from the vertical position and along which the pendulum never becomes horizontal. Hence, we also prove that global stabilization cannot be obtained in the system where the pendulum can impact the horizontal plane (for any mechanical model of impact). Similar results are presented for several analogous systems: a pendulum on a cart, a spherical pendulum, and a pendulum with an additional torque control.

Keywords: stabilization of an inverted pendulum, pendulum on a cart, periodic solution, topological obstructions to stabilization, partial stability

1. Introduction

One and the same property of a system, considered in different contexts, can both be useful, and appear as an undesirable limitation: possible stability of the inverted pendulum to arbitrary horizontal movements of its pivot point [1, 2] turns out to be related to the impossibility of global stabilization of a given position or motion of the pendulum.

The problem of stabilization of the vertical upward position of an inverted pendulum (or of an inverted pendulum on a cart) by means of a horizontal motion on its pivot point (or by a horizontal force, correspondingly) is a well-known problem and has been considered by many authors (see, e.g., [3–8]). This is, among other things, due to the possible applications in real-life systems [9, 10].

It can be shown that the problem of stabilization of the vertical position of an inverted pendulum does not allow continuous autonomous control which would asymptotically lead the pendulum to the vertical from any initial position. This follows from the fact that a continuous function on a circle, which takes values of opposite sign, has at least two zeros, i.e., the system has at least two equilibria (see system (1) below). From topological considerations, it also follows that for the system ‘pendulum on a cart’ (its phase space is $\mathbb{S} \times \mathbb{R}^3$) it is impossible to find such a control that the system has a globally asymptotically stable equilibrium position [4, 11, 12].

The following questions naturally arise. First, do the above statements remain true if we consider the pendulum only in the positions where its mass point is above the pivot point (often there exists a physical constraint in

the system which does not allow the rod to be below the plane of support and it is meaningless to consider the pendulum in such positions). Second, is it true that global stabilization cannot be obtained when the control law is a time-dependent function and it is also a non-periodic function of the position of the pendulum? For a relatively broad class of problems, which may appear in practice, we show that for the both questions the answers are positive.

The proofs are illustrative and based on the Ważewski method [13, 14] and similar to the ones in [1, 2, 15], where the following system has been studied. Let us consider an inverted pendulum in a gravitational field with its pivot point moving along a horizontal line according to a given law of motion. It was proved that, for an arbitrary smooth function, which describes the motion of the pivot point, there always exists a solution such that the pendulum never becomes horizontal along it (never falls). If the law of motion of the pivot point is periodic, then there exists a periodic solution without falling. We add that similar results can be obtained by means of the variational approach [16].

The paper contains two main sections. In one section we consider in detail the case of control of a simple inverted pendulum (system with one degree of freedom), in another section we consider several generalizations of this problem.

2. Simple inverted pendulum

Consider the following control system

$$\begin{aligned}\dot{q} &= p, \\ \dot{p} &= u(q, p, t) \cdot \sin q - \cos q.\end{aligned}\tag{1}$$

Here $u \in \text{Lip}(\mathbb{R}^3, \mathbb{R})$ is the control of the system defined by some Lipschitz function from \mathbb{R}^3 to \mathbb{R} . System (1) describes the motion of a pendulum when the acceleration of its pivot point is given by the function u . The coordinate is chosen so that $q = 0$ and $q = \pi$ correspond to the horizontal positions of the rod, $q = \pi/2$ corresponds to its vertical upward position. Without loss of generality, we assume that the mass of the pendulum, its length and the gravity acceleration equal 1. Note that we do not assume that u is periodic in q .

Suppose that we are looking for a control that would stabilize system (1) in a vicinity of a certain equilibrium position in the following sense. Let M be a subset of the phase space of the system such that the points of M correspond to the positions of the pendulum in which its rod is above the horizontal line (in our case, $M = \{0 < q < \pi\}$) and $\mu \in M$ is the equilibrium for a given control u . We assume that the control function u is chosen in such a way that there exists a closed subset $U \subset M$, $\mu \in U \setminus \partial U$ and a C^1 -function $V: U \rightarrow \mathbb{R}$ with the following properties

- L1. $V(\mu) = 0$ and $V > 0$ in $U \setminus \mu$.
- L2. Derivative \dot{V} with respect to system (1) is negative in $U \setminus \mu$ for all t .

Since the function V can be considered as a Lyapunov function for our system, the equilibrium μ is stable. If the following (stronger) condition holds

- L3. $\dot{V}(x, t) \leq -W(x) < 0$ in $U \setminus \mu$ for all t and $V(0, t) = W(0) = 0$, where $W \in C(U, \mathbb{R})$,

then μ is asymptotically stable. For instance, such a function exists in the following case. Suppose that for a given u , system (1) can be written as follows in a vicinity of μ

$$\dot{x} = Ax + f(x, t),$$

where $x = (q, p)$, A is a constant matrix and its eigenvalues have negative real parts, f is a continuous function and $f(t, x) = o(\|x\|)$ uniformly in t . Then there exists [17] a function V satisfying properties L1, L3.

Theorem 2.1. Let $u(q, p, t) \in \text{Lip}(\mathbb{R}^3, \mathbb{R})$ be a given control function, $\mu \in M$ be an equilibrium for system (1) and $t_0 \in \mathbb{R}$. Suppose there exists a Lyapunov function V satisfying L1 and L2, then there exists an initial condition (q_0, p_0) for $t = t_0$ and an open neighbourhood $B \subset M$ of μ such that, on the interval of existence, the solution $(q(t, q_0, p_0), p(t, q_0, p_0))$ remains in $M \setminus B$.

Proof. For any C^1 function f from \mathbb{R}^n to \mathbb{R} such that $f > 0$ everywhere except one point (where $f = 0$), any level set $f = \varepsilon$, for small $\varepsilon > 0$, is a homotopy sphere [18], and hence a sphere \mathbb{S}^{n-1} .

In our case, for small $\varepsilon > 0$, the set $V = \varepsilon$ is a circle (topologically) in the phase space. We shall denote it by S and the corresponding ball by B . Let us consider the subset $(S \cup \gamma_1 \cup \gamma_2) \times \{t_0\}$ of the extended phase space.

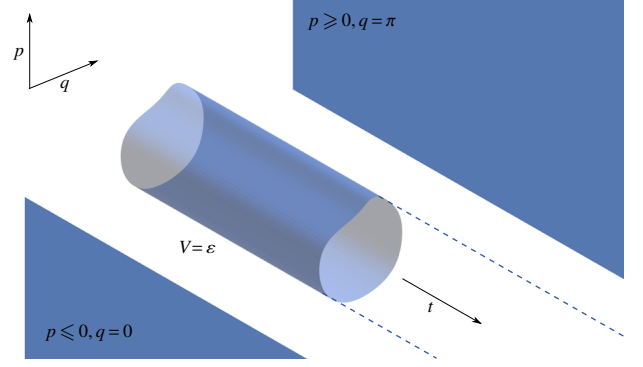


Figure 1: Exit sets for $(M \setminus B) \times \mathbb{R}^+$.

Here, γ_1 is a curve in the phase space which connects S with the set $\{q = 0, p \leq 0\}$. Similarly, γ_2 connects S with the set $\{q = \pi, p \geq 0\}$, $\gamma_1 \cap \gamma_2 = \emptyset$.

Let us introduce the notation $\mathbb{R}^+ = \{t \geq t_0\} \subset \mathbb{R}$. Any solution starting in $M \setminus B$ can leave the set $(M \setminus B) \times \mathbb{R}^+$ only through one of the following sets: $S \times \mathbb{R}^+$, $\{q = 0, p \leq 0\} \times \mathbb{R}^+$ or $\{q = \pi, p \geq 0\} \times \mathbb{R}^+$ (Fig. 1).

Suppose that all solutions starting in $(S \cup \gamma_1 \cup \gamma_2) \times \{t_0\}$ leave $(M \setminus B) \times \mathbb{R}^+$. Then, for every point $(q, p, t_0) \in (S \cup \gamma_1 \cup \gamma_2) \times \{t_0\}$, there is the point of first exit of the corresponding solution from $(M \setminus B) \times \mathbb{R}^+$. This point belongs to one of the above three sets. Therefore, we have a map σ from the set $(S \cup \gamma_1 \cup \gamma_2) \times \{t_0\}$ to the exit set of $(M \setminus B) \times \mathbb{R}^+$. It can be shown that this map is continuous. It follows from the continuous dependence of the solutions on initial conditions and from the fact that solutions of (1) are externally tangent to $(M \cup \partial M) \times \mathbb{R}^+$ at the points where $q = 0$ and $p = 0$ or $q = \pi$ and $p = 0$. Indeed, for any t , if $q = 0$, $p = 0$, then from (1) we find $\dot{p} = \ddot{q} > 0$. The points where $q = \pi$ and $p = 0$ can be considered similarly.

If our assumption is true, then we obtain a continuous map between the connected set $S \cup \gamma_1 \cup \gamma_2$ and a disconnected set $(S, \gamma_1 \cap \partial M \text{ и } \gamma_2 \cap \partial M)$. In order to construct such a function, we can consider compositions of σ with the following maps: the continuous constant maps $\pi_1: \{q = 0, p \leq 0\} \times \mathbb{R}^+ \rightarrow \{\gamma_1 \cap \partial M\} \times \{t_0\}$, $\pi_2: \{q = \pi, p \geq 0\} \times \mathbb{R}^+ \rightarrow \{\gamma_2 \cap \partial M\} \times \{t_0\}$ and the canonical projection $\pi_3: S \times \mathbb{R}^+ \rightarrow S$. The contradiction proves the proposition. \square

From the proof it can be seen that we obtain not a single solution that does not leave the set $M \setminus B$, but a one-parameter family of such solutions. This family can be constructed by varying the paths γ_1 and γ_2 considered in the proof (Fig. 2). In particular, for a given u , there always exists a solution with $p(t_0) = 0$ which are not stabilized.

If we choose the control u so that the solutions can be continued for all t , we can say that there exists a solution that is separated from the equilibrium μ and along this solution the pendulum never becomes horizontal. In particular, suppose we consider our system as a control system with impacts, i.e., we allow the pendulum to fall

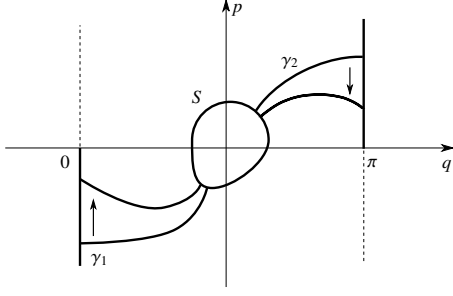


Figure 2: One-parameter family of initial conditions $S \cup \gamma_1 \cup \gamma_2$ can be obtained by varying γ_1 and γ_2 .

on the horizontal plane. From Theorem 2.1, we can conclude that, for any mechanical model of the impact, it is impossible to globally stabilize the rod in a given position.

Note that the solutions can be continued for all t if we assume that $u(q, p, t) \leq a(t)|q| + b(t)|p| + c(t)$, for some continuous functions a , b and c . For instance, if u is bounded (this assumption is natural, since we always have some power limitations), then the solutions exist for all t .

Theorem 2.1 still holds if we consider a more general system

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= u(q, p, t) \cdot \sin q - \cos q + w(q, p, t), \end{aligned} \quad (2)$$

where $w \in \text{Lip}(\mathbb{R}^3, \mathbb{R})$ and $w(0, p, t) < 1$, $w(\pi, p, t) > -1$. Therefore, the system cannot be globally stabilized in the above sense even when there is the control torque w applied at the pivot point.

Some qualitative properties of system (1) can be proved without the assumption on the existence of the function V satisfying L1 and L2. The following result can be proved in essentially the same way as Theorem 2.1.

Theorem 2.2. For any $u(q, p, t) \in \text{Lip}(\mathbb{R}^3, \mathbb{R})$, there exists an initial condition (q_0, p_0) for $t = t_0$ such that on the interval of existence the solution $(q(t, q_0, p_0), p(t, q_0, p_0))$ remains in $\{0 < q < \pi\}$.

In other words, for any control function, there always exists a solution along which the pendulum never falls. Moreover, if we assume that u is a bounded T -periodic function of t and there is a viscous friction in the system, i.e. the dynamics is described by the following equations

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= u(q, p, t) \cdot \sin q - \cos q - \nu p, \end{aligned} \quad (3)$$

then a result, similar to that for an inverted pendulum without control [1, 2], holds

Theorem 2.3. For any bounded and T -periodic in t function $u(q, p, t) \in \text{Lip}(\mathbb{R}^3, \mathbb{R})$ and any $\nu > 0$, there exists an initial condition (q_0, p_0) for $t = 0$ such that the solution $(q(t, q_0, p_0), p(t, q_0, p_0))$ of system (3) is T -periodic and remains in $\{0 < q < \pi\}$ for all t .

Proof. The proof is based on a result from [19, 20], which in our case can be outlined as follows. Let us consider a

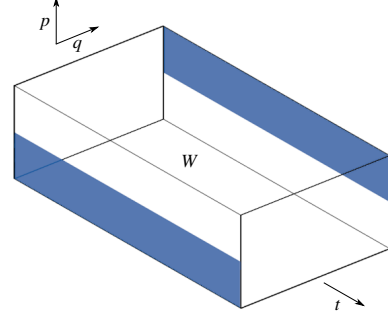


Figure 3: Sets W and W^- (highlighted).

compact subset W of the extended phase space of system (3) of the following form

$$W = \{q, p, t: q \in [0, \pi], p \in [-\rho, \rho], t \in [0, T]\}.$$

If $\rho > 0$ is large, then W and the exit set W^- will be as shown in Fig. 3. Here W^- is as follows

$$\begin{aligned} W^- = & \{q, p, t: q = 0, p \in [-\rho, 0], t \in [0, T]\} \cup \\ & \{q, p, t: q = \pi, p \in [0, \rho], t \in [0, T]\}. \end{aligned}$$

The set W^- is compact and includes all points of exit from the periodic set W for solutions starting in W (formal definition can be found in [19, 20], or, in a simpler form, in [1, 2, 15]). For the set W with the exit set W^- , by direct application of a result of R. Srzednicki [19], it can be obtained that there exists a periodic solution of (3) and this solution remains inside the set W for all t . \square

From Theorems 2.2 and 2.3, we can also conclude that the problem of global stabilization of the pendulum in a position ‘below the horizon’ cannot be solved by means of a Lipschitz control function u .

3. Generalizations and developments

The arguments of the previous section can be carried over to various similar systems. For instance, let us consider the following equations of motion of a controlled inverted spherical pendulum.

$$\begin{aligned} \ddot{\varphi} + u \sin \varphi \cos \theta + v \sin \varphi \sin \theta + \dot{\theta}^2 \cos \varphi \sin \varphi &= -\cos \varphi, \\ \ddot{\theta} \cos^2 \varphi - \dot{\theta} \dot{\varphi} \sin \varphi \cos \varphi + u \sin \theta \cos \varphi - v \cos \varphi \cos \theta &= 0. \end{aligned} \quad (4)$$

Here $\varphi \in (-\pi/2, \pi/2)$ is the inclination angle of the rod, θ is the azimuth angle. Functions $u, v \in \text{Lip}(\mathbb{R}^5, \mathbb{R})$, $u = u(\varphi, \dot{\varphi}, \theta, \dot{\theta}, t)$, $v = v(\varphi, \dot{\varphi}, \theta, \dot{\theta}, t)$ are the control accelerations of the pivot point (projections of the acceleration on fixed axes in the horizontal plane). We use the same assumptions concerning the mass and the length of the pendulum and the gravity acceleration as in the previous section.

The configuration space of the system is a two-dimensional sphere. Let M be a subset of the phase space such that the points of M correspond to the positions of the pendulum in which its rod is above the horizontal plane ($\varphi = 0$). Let us suppose that the control functions u, v are chosen in such a way that $\mu \in M$ is an equilibrium of system (4) and, in a vicinity of μ , there exists a Lyapunov function satisfying L1 and L2. Then global stabilization cannot be achieved for the system. To be more precise, the following holds.

Theorem 3.1. Let $u, v \in \text{Lip}(\mathbb{R}^5, \mathbb{R})$ be given control functions, $\mu \in M$ be an equilibrium for system (4) and $t_0 \in \mathbb{R}$. Suppose there exists a Lyapunov function V satisfying L1 and L2, then there exists an initial condition $(\varphi_0, \dot{\varphi}_0, \theta_0, \dot{\theta}_0)$ for $t = t_0$ and an open neighbourhood $B \subset M$ of μ such that on the interval of existence the solution $(q(t, q_0, p_0), p(t, q_0, p_0))$ remains in $M \setminus B$.

Proof. The main idea of the proof is similar to the one in Theorem 2.1. The only difference is that it is sufficient to connect the sphere S (level set $V = \varepsilon$ for small $\varepsilon > 0$) by one curve γ with the set $\{\varphi = 0, \dot{\varphi} \leq 0\}$.

Now, if we assume that all solutions starting in $\gamma \times \{t_0\}$ leave $M \setminus B$, then we obtain a continuous map between a connected set and its two-point boundary ($\gamma \cap S$ and $\gamma \cap \{\varphi = 0, \dot{\varphi} \leq 0\}$).

The continuity of the corresponding map follows, as in Theorem 2.1, from the fact that if we put the rod in the horizontal position and $\dot{\varphi} = 0$, then for any control functions, the pendulum will move to the region where $\varphi < 0$. In other words, if some solution leaves $M \setminus B$, then all close solution also leave this set. \square

From the proof it can also be seen that we obtain not a single solution, that does not leave the set $M \setminus B$, but a three-parameter family of such solutions: two point in a four-dimensional space can be connected by a three-dimensional family of paths.

The most important generalization of the system considered in the previous section is the controlled system of a pendulum on a cart, which is more correct from the physical point of view than its limiting case, a simple controlled inverted pendulum.

The equations of motion of a pendulum on a cart have the following form

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= \frac{u(q, p, x, y, t) \sin q + p^2 \sin q \cos q - (1 + m) \cos q}{m + \cos^2 q}, \\ \dot{x} &= y, \\ \dot{y} &= (m + \cos^2 q)^{-1} (u(q, p, x, y, t) + p^2 \cos q - \sin q \cos q). \end{aligned} \quad (5)$$

Here $m > 0$ is the mass of the cart, x is the coordinate of the pivot point on the horizontal line, $u \in \text{Lip}(\mathbb{R}^5, \mathbb{R})$ is the horizontal force applied to the cart. We assume

that the mass of the pendulum, its length and the gravity acceleration equal 1.

Note that if the control u does not depend on the position and velocity of the pivot point (x and y , correspondingly), then the first two equations can be considered separately. Nonetheless, we will consider the general case, when the control function u is non-autonomous and may depend on the total angular distance covered by the rod q (again, we do not assume that u is periodic in q), on the angular velocity of the rod p and on the variables x and y , defined above.

Theorem 3.2. Let $u \in \text{Lip}(\mathbb{R}^5, \mathbb{R})$ be a given control function, $\mu \in M$ be an equilibrium for system (5) and $t_0 \in \mathbb{R}$. Suppose there exists a Lyapunov function V satisfying L1 and L2, then there exists an initial condition (q_0, p_0, x_0, y_0) for $t = t_0$ and an open neighbourhood $B \subset M$ of μ such that the solution starting at (q_0, p_0, x_0, y_0) remains in $M \setminus B$ on the interval of existence.

Here, as usual, by M we denote the points of the phase space where $0 < q < \pi$. The proof is similar to the case of Theorem 2.1. Note again that we obtain a three-parameter family of the solutions, not a single solution.

Everywhere above, we have never fully used the fact that V is a Lyapunov function satisfying L1 and L2 (even though, in real-life applications, it is quite natural to assume the existence of such a function). Indeed, what we use is that there exists a ‘capturing neighbourhood’ of some point.

If we omit the requirement concerning the existence of a Lyapunov function and do not require the stability in all variables, then, similarly to the theorems above, we can prove

Theorem 3.3. Let $q_{eq} \in (0, \pi)$ and $t_0 \in \mathbb{R}$. Suppose that $u(q, p, x, y, t) \in \text{Lip}(\mathbb{R}^5, \mathbb{R})$ and $Z = B \times \mathbb{R}^2 \subset M$ is a closed cylinder such that ∂B is a Jordan curve in the plane (q, p) , $(q_{eq}, 0) \in B$ and any solution of system (5) starting in ∂Z at $t \geq t_0$ locally belongs to $Z \setminus \partial Z$. Then there exists a solution which remains in $M \setminus Z$ on the interval of existence.

In this statement we do not require the stability in the variables x and y . We only assume that any solution starting in a neighborhood of the point $(q_{eq}, 0)$, will stay in this neighborhood w.r.t. to the variables q and p . In particular, we do not assume that this point is a stable equilibrium.

In contrast, if we assume that the control function u is T -periodic in t , then, taking into account the results on the existence of a Lyapunov function in a vicinity of an asymptotically stable equilibrium, it may be useful to consider more complex ‘capturing sets’. In particular, let us consider system (5) and suppose that its right-hand side is T -periodic in t . Suppose that $\mu \in M$ is an asymptotically stable equilibrium. Then there exists a smooth T -periodic Lyapunov function $V(q, p, x, y, t)$ satisfying the conditions of the Lyapunov theorem on asymptotic stability [21]. By means of the function V , the following result can be proved

Theorem 3.4. Let $u \in \text{Lip}(\mathbb{R}^5, \mathbb{R})$ be a given T -periodic function, $\mu \in M$ be an asymptotically stable equilibrium for system (5) and $t_0 \in \mathbb{R}$. Then there exists an initial condition (q_0, p_0, x_0, y_0) for $t = t_0$ and an open neighbourhood $B \subset M$ of μ such that the solution starting at (q_0, p_0, x_0, y_0) remains in $M \setminus B$ on the interval of existence.

The proof is based on the consideration of the T -periodic level set $V = \varepsilon$ of the Lyapunov function, for small $\varepsilon > 0$. The function V is defined in a vicinity of the point μ . Any solution starting in the set $V = \varepsilon$ belongs to the set $V < \varepsilon$ for all subsequent t . Therefore, the proof can be obtained by using similar arguments as in Theorem 2.1.

4. Conclusion

We have shown that the systems considered in the paper cannot be globally stabilized, provided some natural assumptions on the control law. Similar results can be proved for the case when we try to stabilize the system in a vicinity of a given trajectory, which may not be an equilibrium. Also, we can consider systems with dry or viscous friction and systems different from the inverted pendulum. For instance, we can consider the control system of a point moving on a surface which intersects the horizontal plane orthogonally. In all these and many other cases, if the solutions depend continuously on initial data, the same topological obstructions to global stabilization appear and the above methods can be applied.

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